Good Synchronization Sequences for Permutation Codes

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APPENDIX

A. Definition of the Function $F_k$ in $P_{EAD}$

$F_k$ is a recursive function defined as

$$F_k = \begin{cases} 
0, & k = 1 \\
1, & k = 2 \\
(k-1)[F_{k-2} + F_{k-1}], & k > 2
\end{cases}$$

B. Proof of the Good Synchronization Sequences Algorithm

Proof: Due to the shifting operation on $P(= S^0_C)$, every $S_C$, $i = 0 \ldots M - 1$, is related to $P$ as follows: Let $X^i$ and $Y^i$ be subsequences of the $i$ and $M - i$ consecutive digits of $P$, respectively, such that $X^i = P_1 \ldots P_i$ and $Y^i = P_{i+1} \ldots P_M$, then $P = X^i Y^i$ and $S_C = Y^i X^i$.

We have shown that for $P$ to be formed we need $X^i$ and $Y^i$ concatenated as $X^i Y^i$ for $i = 0 \ldots M - 1$ or as $Y^i X^i$ for $i = 0$, and that can only happen, in the absence of $P$ itself, as follows: Consider any two permutations $P''$, $P''' \in S_M \setminus \{P\}$ concatenated as $P'' P'''$, $i = 0, \ldots, M - 1$, or as $P''' P''$, for $i = 0$, the concatenation of $P''$ and $P'''$ can only form $P$ if, $P'' = P_{i+1} \ldots P_M X^i$ and $P''' = Y^i P''_1 \ldots P''_i$.

$$P'' P''' = P''_{i+1} \ldots P''_M X^i Y^i P'''_{i+1} \ldots P'''_i$$

$$= P''_{i+1} \ldots P''_M P P'''_{i+1} \ldots P'''_i \quad \text{(1)}$$

Note:
- We included the concatenation $P''' P''$, for $i = 0$, for completeness, otherwise all cases are covered by the concatenation $P'' P'''$, for $i = 0, \ldots, M - 1$.
- For the case $P'' = P'''$, $P'' = P''' = Y^i X^i$, we get all the $M$ cyclic shifts of $P, S_C$, where $i = 0, \ldots, M - 1$.

The RHS of (1) tells us the following: For $i = 0, \ldots, M - 1$,
- there are $M - i$ and $i$ digits on the left and right of the sequence $\ldots X^i Y^i \ldots$, respectively.
- there is a total of $(M - i)!$ and $i!$ possible permutations of the form $P''$ and $P'''$, respectively, and any combination of these will produce $P$ when concatenated as $P'' P'''$.

To avoid the sequence $\ldots X^i Y^i \ldots$ (producing $P$) in the concatenation $P'' P'''$, we need to remove all the $(M - i)!$ possible permutations of the form $P''$ or all the $i!$ permutation of the form $P'''$ from $S_M$ and include them in $S_E$. This has to be done such that only a minimal number of permutations are included in $S_E$ (or removed from $S_M$), as follows: For $i = 0, \ldots, M - 1$,

1) include all $P'''$ in $S_E$ if: $i < (M - i), i < M/2$.
2) include $(M - i)! P'''$ in $S_E$ if: $i > (M - i), i > M/2$.
3) include either all $P''$ or $(M - i)! P'''$ in $S_E$: $i = (M - i), i = M/2$.

Next, we write the conditions of steps 1)-3) in terms of $m$ for both even and odd $M$.

For odd $M$: $M = 2m + 1$,

1) $i < (2m + 1 - i), i < (2m + 1)/2, i < (m + 1/2), i \leq m$
2) $i > (2m + 1 - i), i > (2m + 1)/2, i > (m + 1/2), i \geq m + 1$
3) $i = (2m + 1 - i), i = (2m + 1)/2, i = (m + 1/2)$, not possible because $i$ and $m$ are integers.

$$\sigma_{\text{odd}} = \sum_{i=0}^{m} i! + \sum_{i=m+1}^{2m} (2m + 1 - i)!$$

$$= 1 + \sum_{i=1}^{m} i! + \sum_{i=m+1}^{2m} (2m + 1 - i)!,$$

the two summations each yields the same result, hence we can reduce the equation to

$$\sigma_{\text{odd}} = 1 + 2 \sum_{i=1}^{m} i!.$$

For even $M$: $M = 2m$,

1) $i < (2m - i), i < (2m)/2, i < m$
2) $i > (2m - i), i > (2m)/2, i > m$
3) $i = (2m - i), i = (2m)/2, i = m$.

$$\sigma_{\text{even}} = \sum_{i=0}^{m} i! + \sum_{i=m+1}^{2m} (2m + 1 - i)!$$

$$= 1 + \sum_{i=1}^{m} i! + \sum_{i=m+1}^{2m} (2m + 1 - i)!,$$

the two summations each yields the same result, hence we can reduce the equation to

$$\sigma_{\text{even}} = 1 + 2 \sum_{i=1}^{m} i!.$$

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\[ \sigma_{\text{even}} = \sum_{i=0}^{m} i! + \sum_{i=m+1}^{2m-1} (2m-i)! \]
\[ = 1 + m! + \sum_{i=1}^{m-1} i! + \sum_{i=m+1}^{2m-1} (2m-i)!, \]

the two summations each yields the same result, hence we can reduce the equation to

\[ \sigma_{\text{even}} = 1 + m! + 2 \sum_{i=1}^{m-1} i!. \]

It can be seen that \(|S|\) on the RHS of (1) simply means that for every permutation added in \(S\) there are \(\sigma\) permutations added to \(S_E\).

\[ \blacksquare \]

\textit{C. Codebooks used in the simulations}

\[ C_1 = \left\{ 1234, 1243, 1342, 1423, 2134, 2314, 2341, 2431, 3124, 3142, 3421, 3412, 4123, 4213, 4231, 4312 \right\}. \]

\[ C_2 = \left\{ 23451, 34512, 45123, 23541, 35412, 54123, 24513, 45132, 32541, 25413, 54132, 13452, 34521, 45213, 13542, 35421, 54213, 31452, 14523, 31542, 45231, 12453, 24531, 45312, 12543, 25431, 54312, 21453, 21543 \right\}. \]